Interactions between Combinatorics, Geometry and Harmonic Analysis

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1 Geometry of projections and the Sobolev inequality

Theorem 1.1. Loomis-Whitney Inequality
If $\|\pi_j(X)\| \leq A$, then $\|X\| \lesssim A^{\frac{n}{n-1}}$, where X is a set of unit cubes in the unit cubical lattice in $\mathbb{R}^n$, $\|X\|$ is its volume, and $\pi_j(X)$ is the projection onto the coordinate hyperplane perpendicular to the x_j-axis.

Sketch of proof: We show that if $\|\pi_j(X)\| \leq B$ for every j, then there exists a column of cubes with between 1 and $B^{\frac{1}{n-1}}$ cubes of X, and then we use induction.

Theorem 1.2. Generalization of Loomis-Whitney Inequality
If U is open set in $\mathbb{R}^n$ with $\|\pi_j(U)\| \leq A$, then $\|U\| \lesssim A^{\frac{n}{n-1}}$.

Sketch of proof: take $U_\varepsilon \subset U$, which is the biggest union of $\varepsilon-$cubes in $\varepsilon-$lattice.

Corollary 1.3. Isoperimetric inequality
If U is a bounded open set in $\mathbb{R}^n$, then $\mathrm{Vol}_n(U) \lesssim \mathrm{Vol}_{n-1}(\partial U)^{\frac{n}{n-1}}$.

Sketch of proof: Using previous theorem we get $\mathrm{Vol}_n(U) \lesssim (\max_j |\pi_j(U)|)^{\frac{n}{n-1}} \leq \mathrm{Vol}_{n-1}(\partial U)^{\frac{n}{n-1}}$.

2 Sobolev Inequality

$$S_u(h) \coloneqq \{x \in \mathbb{R}^n \text{ so that } |u(x)| > h\}.$$

Lemma 2.1. If $u \in C^1_{\mathrm{comp}}(\mathbb{R}^n)$, then for any j, $|\pi_j(S_u(h))| \leq h^{-1} \cdot \|\nabla u\|_{L^1}$.

$$T_u(k) \coloneqq \{x \in \mathbb{R}^n \text{ so that } 2^k < |u(x)| \leq 2^{k+1}\}.$$

Lemma 2.2. If $u \in C^1_{\mathrm{comp}}(\mathbb{R}^n)$, then for any j, $|\pi_j T_u(k)| \lesssim 2^{-k} \int_{T_u(k-1)} |\nabla u|$.

Theorem 2.3. Sobolev inequality
If $u \in C^1_{\mathrm{comp}}(\mathbb{R}^n)$, then $\|u\|_{L^{\frac{n}{n-1}}} \lesssim \|\nabla u\|_{L^1}$.

Sketch of proof: From Loomis-Whitney, theorem 1.1, theorem 1.2 and previous lemma we get

$$|T_u(k)| \lesssim 2^{-k\frac{n}{n-1}} \left(\int_{T_u(k-1)} |\nabla u| \right)^{\frac{n}{n-1}},$$

and therefor

$$\int |u|^{\frac{n}{n-1}} \sim \sum_{k\in\mathbb{Z}} |T_u(k)| 2^{k\frac{n}{n-1}} \lesssim \sum_{k\in\mathbb{Z}} \left(\int_{T_u(k-1)} |\nabla u| \right)^{\frac{n}{n-1}} \leq \left(\int_{\mathbb{R}^n} |\nabla u| \right)^{\frac{n}{n-1}}.$$

3 Intersection patterns of balls in Euclidean space

Lemma 3.1. Vitali Covering Lemma
If $\{B_i\}_{i\in I}$ is a finite collection of balls in $\mathbb{R}^n$, then there exists a subcollection $J \subset I$ such that $\{B_j\}_{j\in J}$ are disjoint but $\bigcup_{i\in I} B_i \subset \bigcup_{j\in J} 3B_j$.

Sketch of proof: We choose a subset of disjoint balls such that their radii are big enough. Then we show that every ball intersects with some from our subset.

Lemma 3.2. Ball doubling
If $\{B_i\}_{i\in I}$ is a finite collection of balls, then $|\bigcup 2B_i \leq 6^n |\bigcup B_i|$.

Sketch of proof: We use Vitali Covering Lemma for $2B_i$ and notice the relations between the volumes of the sets.

Theorem 3.3. Vitali Covering Lemma for infinite collections of balls
Suppose $\{B_i\}_{i\in I}$ is a finite collection of balls in $\mathbb{R}^n$, and there exist finite constant M such that any disjoint subset of the balls $\{B_i\}_{i\in I}$ has total volume at most M. Then there exists a subcollection $J \subset I$ such that $\{B_j\}_{j\in J}$ are disjoint but $\bigcup_{i\in I} B_i \subset \bigcup_{j\in J} 4B_j$.

Sketch of proof: We repeat proof of Vitali Covering Lemma but with taking for our collections balls with radii that are big enough (3/4 of supremal radius) instead of maximal ones (which may not exist).

4 Hardy-Littlewood maximal function

Definition 4.1. Average of a function f on a set A
$\fint_A f := \frac{1}{\mathrm{Vol}A} \int_A f$.

Definition 4.2. Hardy-Littlewood maximal function
$Mf(x) := \sup_r \fint_{B(x,r)} |f|$.

Lemma 4.3. For each $h > 0$, $|S_{Mf}(h)| \lesssim h^{-1} \|f\|_{L^1}$.

Sketch of proof: We observe that we can cover $S_{Mf}(h)$ with balls and we use Vitali Covering Lemma for infinite collections of balls.

Lemma 4.4. $|S_{Mf}(h)| \lesssim h^{-1} \int_{S_f(h/2)} |f|$.

Sketch of proof: We use balls from previous proof and observe that $\int_{B_j \cap S_f(h/2)} |f| \geq \frac{h}{2}|B_j|$.

Theorem 4.5. Hardy-Littlewood
For any dimension n and any $p > 1$, there is a constant $C(n,p)$ so that $\|Mf\|_{L^p(\mathbb{R}^n)} \leq C(n,p)\|f\|_{L^p(\mathbb{R}^n)}$.

Sketch of proof: We use previous lemma and then observe the behaviour of geometric sum.

5 L^p estimates for linear operators

Proposition 5.1. Suppose that T obeys the inequality $\|Tf\|_{L^q(\mathbf{R}^n)} \leq C\|f\|_{L^p}$. If the measure of the support of f is equal to V, and if $|f| \leq h$ everywhere, then $|S_{Tf}(H)| \leq C^q V^{q/p}(h/H)^q$.

Definition 5.2. Convolution
$f, g : \mathbb{R}^n \to \mathbb{R}$,
$(f * g)(x) := \int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy$.

$$T_\alpha f := f * |x|^{-\alpha} = \int_{\mathbb{R}^n} f(y)|x-y|^{-\alpha}dy, \text{ where } 0 < \alpha < n.$$

Proposition 5.3. Fix a dimension n and consider the linear operator T_α. The following are equivalent:

1. There exists a constant C so that for every $r > 0$, $\|T_{\alpha\chi_{B_r}\|_q} \leq C\|\chi_{B_r}\|_p$.
2. $p > 1$ and $\alpha = n(1 - \frac{1}{q} + \frac{1}{p})$.

Theorem 5.4. Hardy-Littlewood-Sobolev
If $p > 1$ and $\alpha = n(1 - \frac{1}{q} + \frac{1}{p})$, then $\|T_\alpha f\|_q \leq C(n,p,q)\|f\|_p$.

6 Proof of the Hardy-Littlewood-Sobolev inequality

Lemma 6.1. $T_\alpha f(x) = \int_0^\infty r^{n-\alpha-1} \left(\oint_{B(x,r)} f \right) dr$

Sketch of proof of the Hardy-Littlewood-Sobolev inequality: We use both definition of $Mf(x)$ and Holder's inequality to get upper bounds for $\oint_{B(x,r)} f$, and then previous lemma. By choosing right constants depending on α, p, n we then apply Hardy and Littlewood theorem.